HIROTA BILINEAR FORMALISM AND SUPERSYMMETRY

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Abstract

Extending the gauge-invariance principle for τ functions of the standard bilinear formalism to the supersymmetric case, we define $\mathcal{N}=1$ supersymmetric Hirota operators. Using them, we bilinearize SUSY nonlinear evolution equations. The super-soliton solutions and extension to SUSY sine-Gordon are also discussed. As a quite strange paradox it is shown that the Lax integrable SUSY KdV of Manin-Radul-Mathieu equation does not possess N super-soliton solution for $N \geq 3$ for arbitrary parameters. Only for a particular choice of them the N super-soliton solution exists.

I. INTRODUCTION

Supersymmetric integrable systems constitute a very interesting subject and as a consequence a number of well known integrable equations have been generalized into supersymmetric (SUSY) context. We mention the SUSY versions of sine-Gordon [1], [2], KP-hierarchy [3], KdV [3], [4], Boussinesq [6] etc. We also point out that there are many generalizations related to the number $\mathcal{N}$ of fermionic independent variables. In this paper we are dealing with the $\mathcal{N}=1$ superspace.

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So far, many of the tools used in standard theory have been extended to this framework, such as Bäcklund transformations [2], prolongation theory, hamiltonian formalism [7], grasmmannian description [8], τ functions [9], Darboux transformations [10]. The physical interest in the study of these systems have been launched by the seminal paper of Alvarez-Gaume et. al [11] about the partition function and super-Virasoro constraints of 2D quantum supergravity. Although the τ function theory in the context of SUSY pseudodifferential operators was given for the SUSY KP-hierarchy [8], the bilinear formalism for SUSY equations was very little investigated. We mention here the algebraic approach using the representation theory of affine Lie super-algebras in the papers of Kac and van der Leur [12], Kac and Medina [13] the super-conformal field theoretic approach of LeClair [14]. Anyway in these articles the bilinear hierarchies are not related to the SUSY hierarchies of nonlinear equations.

This paper which is an extended version of [16] we consider a direct approach to SUSY equations in a $\mathcal{N} = 1$ superspace rather than hierarchies namely extending the gauge-invariance principle of τ functions for classical Hirota operators. Our result generalize the Grammaticos-Ramani-Hietarinta [17] theorem, to SUSY case and we find $\mathcal{N} = 1$ SUSY Hirota bilinear operators. With these operators one can obtain SUSY-bilinear forms for SUSY KdV equation of Mathieu [4] and also it allows bilinear forms for certain SUSY extensions of Sawada-Kotera-Ramani [18], Hirota-Satsuma [19], KdV-B [5], Burgers, mKdV and $\mathcal{N} = 2$ -superspace SUSY sine-Gordon equations [25]. Also the gauge-invariance principle allows to study the SUSY multisoliton solutions as exponentials of linears. A very interesting fact which happens is that SUSY KdV equation of Mathieu does not have 3 supersoliton solution for arbitrary choice of solitary waves although it possesses Lax pair [4]. Only for special combination of parameters the equation admits N soliton solutions. Although this seems to be a quite strange paradox, Liu and Manas [10], found also super-soliton solutions for SUSY KdV equation in terms of pfaffians only for certain wave parameters. This fact shows that Hirota integrability and Lax integrability are different in the SUSY context.

The paper is organized as follows. In section II the standard bilinear formalism is briefly

discussed. In section III supersymmetric versions for nonlinear evolution equations are presented and in section IV we introduce the super-bilinear formalism. In the last section we shall present the bilinear form for SUSY KdV-type equations, super-soliton solutions and several comments about extension to $\mathcal{N} = 2$ SUSY sine-Gordon equation.

II. STANDARD BILINEAR FORMALISM

The Hirota bilinear operators were introduced as an antisymmetric extension of the usual derivative [21], because of their usefulness for the computation of multisoliton solution of nonlinear evolution equations. The bilinear operator $\mathbf{D}_x = \partial_{x_1} - \partial_{x_2}$, acts on a pair of functions (the so-called "dot product") antisymmetrically:

$$\mathbf{D}_x f \bullet g = (\partial_{x_1} - \partial_{x_2}) f(x_1) f(x_2)|_{x_1 = x_2 = x} = f'g - fg'. \tag{2.1}$$

The Hirota bilinear formalism has been instrumental in the derivation of the multisoliton solutions of (integrable) nonlinear evolution equations. The first step in the application is a dependent variable transformation which converts the nonlinear equation into a quadratic form. This quadratic form turns out to have the same structure as the dispersion relation of the linearized nonlinear equation, although there is no deep reason for that. This is best understood if we consider an example. Starting from paradigmatic KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.2}$$

we introduce the substitution $u = 2\partial_x^2 \log F$ and obtain after one integration:

$$F_{xt}F - F_x F_t + F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 = 0, \tag{2.3}$$

which can be written in the following condensed form:

$$(\mathbf{D}_x \mathbf{D}_t + \mathbf{D}_x^4) F \bullet F = 0. \tag{2.4}$$

The power of the bilinear formalism lies in the fact that for multisoliton solution F's are polynomials of exponentials. Moreover it displays also the interaction (phase-shifts) between solitons. In the case of KdV equation the multisoliton solution has the following form:

$$F = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} A_{ij}\mu_i\mu_j\right), \tag{2.5}$$

where $\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}$ and $expA_{ij} = (\frac{k_i - k_j}{k_i + k_j})^2$ which is the phase-shift from the interaction of the soliton "i" with the soliton "j".

This picture can be generalized to any bilinear equation of the form

$$P(\mathbf{D}_{\vec{x}})F \bullet F = 0 \tag{2.6}$$

where P is any polynomial and $\vec{x} = (t, x, y, ...)$. The 1 soliton solution is $F = 1 + e^{\eta}$, where $\eta = \vec{k}\vec{x} + const$. This solution holds if

$$P(\vec{k}) = 0.$$

This is a condition on the parameters $\vec{k}$ of η and is called the *dispersion relation.* If the parameter space is n-dimensional then the above equation defines an $n-1$ dimensional submanifold called *dispersion manifold.*

Hirota ansatz for 2 soliton solution is

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}$$

where η's are defined before and A_{12} are function of $\vec{k}_1$ and $\vec{k}_2$ each one giving the coordinates of some point in the dispersion manifold. Substituting the ansatz in the equation (2.6) and taking into account the dispersion relation we find

$$A_{12}P(\vec{k}_1 + \vec{k}_2) + P(\vec{k}_1 - \vec{k}_2) = 0. \tag{2.7}$$

Generally speaking, the great majority of bilinear equations possess 2 soliton solution but only *integrable* equations possess 3 or more soliton solutions. The form of the N soliton solution is,

$$F = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} A_{ij}\mu_i\mu_j\right)$$

More about the bilinear equations are in [22]

A very important observation (which motivated the present paper) is the relation of the physical field $u = 2\partial_x^2 \log F$ of KdV equation with the Hirota function F: the gauge-transformation $F \to e^{px+\omega t}F$ leaves u invariant. This is a general property of all bilinear equations. Moreover, one can define the Hirota operators using the requirement of gauge-invariance. Let's introduce a general bilinear expression,

$$A_N(f,g) = \sum_{i=0}^{N} c_i (\partial_x^{N-i} f)(\partial_x^i g) \tag{2.8}$$

and ask to be invariant under the gauge-trasformation:

$$A_N(e^\theta f, e^\theta g) = e^{2\theta} A_N(f,g) \quad \theta = kx + \omega t + ...(linears). \tag{2.9}$$

Then we have the following, [17]

Theorem: $A_N(f,g)$ *is gauge-invariant if and only if* $A_N(f,g) = \mathbf{D}_x^N f \bullet g$ i.e.

$$c_i = c_0(-1)^i \begin{pmatrix} N \\ i \end{pmatrix}$$

and c_0 is a constant and the brakets represent binomial coefficient.

We must point out that in the whole paper the natural number N (which will denote number of solitons, number of terms, exponents etc.) is different from $\mathcal{N}$ which is related to supersymmetry or superspace.

III. SUPERSYMMETRY

The supersymmetric extension of a nonlinear evolution equation (KdV for instance) refers to a system of coupled equations for a bosonic $u(t,x)$ and a fermionic field $\xi(t,x)$ which reduces to the initial equation in the limit where the fermionic field is zero (bosonic limit). In the classical context, a fermionic field is described by an anticommuting function with values in an *infinitely* generated Grassmann algebra. However, supersymmetry is not just a coupling of a bosonic field to a fermionic field. It implies a transformation (supersymmetry invariance) relating these two fields which leaves the system invariant. In order to have a

mathematical formulation of these concepts we have to extend the classical space (x, t) to a larger space (superspace) (t, x, θ) where θ is a Grassmann variable and also to extend the pair of fields (u, ξ) to a larger fermionic or bosonic superfield $\Phi(t, x, \theta)$. In order to have nontrivial extension for KdV we choose Φ to be fermionic, having the expansion

$$\Phi(t, x, \theta) = \xi(t, x) + \theta u(t, x). \tag{3.1}$$

The $\mathcal{N} = 1$ SUSY means that we have only one Grassmann variable θ and we consider only space supersymmetry invariance namely $x \to x - \lambda\theta$ and $\theta \to \theta + \lambda$ (λ is an anti-commuting parameter). This transformation is generated by the operator $Q = \partial_\theta - \theta\partial_x$, which anticommutes with the covariant derivative $D = \partial_\theta + \theta\partial_x$ (Notice also that $D^2 = \partial_x$). Expressions written in terms of the covariant derivative and the superfield Φ are manifestly supersymmetric invariant. Using the superspace formalism one can construct different supersymmetric extension of nonlinear equations. Thus the integrable (in the sense of Lax pair) variant of $\mathcal{N} = 1$ SUSY KdV is [3] [4]

$$\Phi_t + D^6\Phi + 3D^2(\Phi D\Phi) = 0, \tag{3.2}$$

which on the components has the form

$$\begin{aligned} u_t &= -u_{xxx} - 6uu_x + 3\xi\xi_{xx} \\ \xi_t &= -\xi_{xxx} - 3\xi_x u - 3\xi u_x. \end{aligned} \tag{3.3}$$

Another integrable variant of SUSY KdV equation which is very important in applications to supersymmetric matrix models is SUSY KdV-B equation [5], namely

$$\Phi_t + D^6\Phi + 6D^2\Phi D\Phi = 0, \tag{3.4}$$

leads to a somewhat trivial system in which the fermionic fields decouple from the bosonic equation which reduces then to the usual KdV.

We shall discuss also the following supersymmetric equations, although we do not know if it is completely integrable in the sense of Lax pair (Φ is also a fermionic superfield).

- $\mathcal{N}=1$ SUSY Sawada-Kotera-Ramani,

$$\Phi_t + D^{10}\Phi + D^2(10D\Phi D^4\Phi + 5D^5\Phi\Phi + 15(D\Phi)^2\Phi) = 0. \tag{3.5}$$

- $\mathcal{N}=1$ SUSY Hirota-Satsuma (shallow water wave)

$$D^4\Phi_t + \Phi_t D^3\Phi + 2D^2\Phi D\Phi_t - D^2\Phi - \Phi_t = 0 \tag{3.6}$$

- $\mathcal{N}=1$ SUSY Burgers

$$\Phi_t + \Phi D\Phi_x + \Phi_{xx} = 0 \tag{3.7}$$

A very important equation from the physical consideration is the SUSY sine-Gordon. We are going to consider the version studied by Kulish and Tsyplyaev [25]. There are other integrable versions of SUSY sine-Gordon emerged from algebraic procedures [24]. In this case one needs two Grassmann variables θ_α with $\alpha = 1, 2$ and the supersymmetry transformation is

$$x'^\mu = x^\mu - i\bar{\lambda}\gamma^\mu\theta, \quad \theta'_\alpha = \theta_\alpha + \lambda_\alpha, \mu = 1, 2.$$

Here, λ_α is the anticommuting spinor parameter of the transformation and $\bar{\lambda} = (\lambda^1, \lambda^2)$, $\lambda^\alpha = \lambda_\beta(i\sigma_2)^{\beta\alpha}$, $\gamma^0 = i\sigma_2$, $\gamma^1 = \sigma_1$, $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$. We use the metric $g^{\mu\nu} = diag(-1, 1)$ and σ_i are the Pauli matrices. The superfield has the following expansion:

$$\Phi(x^\mu, \theta_\alpha) = \phi(x^\mu) + i\bar{\theta}\psi(x^\mu) + \frac{i}{2}\bar{\theta}\theta F(x^\mu), \tag{3.8}$$

where ϕ and F are real bosonic (even) scalar fields and ψ_α is a Majorana spinor field. The SUSY sine-Gordon equation is:

$$\bar{D}D\Phi = 2i\sin\Phi, \tag{3.9}$$

where $D_\alpha = \partial_{\theta^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu$ and on the components it has the form:

$$\begin{gathered}(\gamma^\mu\partial_\mu + \cos\phi)\psi = 0 \\ \phi_{xx} - \phi_{tt} = \frac{1}{2}(\sin(2\phi) - i\bar{\psi}\psi\sin\phi).\end{gathered} \tag{3.10}$$

IV. SUPER-HIROTA OPERATORS

In order to apply the bilinear formalism on these equations one has to define a SUSY bilinear operator. We are going to consider the following general $\mathcal{N}=1$ SUSY bilinear expression

$$S_N(f,g)=\sum_{i=0}^{N} c_i(D^{N-i}f)(D^i g), \tag{4.1}$$

for any N, where D is the covariant derivative and f, g are Grassmann valued functions (odd or even). We shall prove the following

Theorem: The general $\mathcal{N}=1$ SUSY bilinear expression (4.1) is super-gauge invariant i.e. for $\Theta = kx+\omega t+\theta\hat{\zeta}+$...linears ($\zeta$ is a Grassmann parameter)

$$S_N(e^{\Theta}f, e^{\Theta}g)=e^{2\Theta}S_N(f,g),$$

if and only if

$$c_i=c_0(-1)^{i|f|+\frac{i(i+1)}{2}}\begin{bmatrix} N \\ i \end{bmatrix},$$

where the super-binomial coefficients are defined by:

$$\begin{bmatrix} N \\ i \end{bmatrix}=\begin{cases} \begin{pmatrix} \text{[N/2]} \\ \text{[i/2]} \end{pmatrix} & \text{if } (N,i)\neq(0,1)mod2 \\ 0 & \text{otherwise} \end{cases}$$

$|f|$ is the Grassmann parity of the function f defined by:

$$|f|=\begin{cases} 1 \ \text{ if } f \text{ is odd (fermionic)} \\ 0 \ \text{ if } f \text{ is even (bosonic)} \end{cases}$$

and $[k]$ is the integer part of the real number k ($[k]\leq k<[k]+1$)

Proof: First we are going to consider N even and we shall take it on the form $N=2P$. In this case we have:

$$S_N(f,g)=\sum_{i=1}^{N} c_i(D^{N-i}f)(D^i g)=\sum_{i=0}^{P} c_{2i}(\partial^{P-i}f)(\partial^i g)+\sum_{j=0}^{P-1} c_{2j+1}(\partial^{P-j-1}Df)(\partial^j Dg)$$

Imposing the super-gauge invariance and expanding the covariant derivatives we obtain:

$$\sum_{n\geq 0}\sum_{m\geq 0}\left(\sum_{i=0}^{P} c_{2i}\begin{pmatrix} i \\ n \end{pmatrix}\begin{pmatrix} P-i \\ m \end{pmatrix} k^{P-n-m}\right)(\partial^m f)(\partial^n g)+$$

$$+\sum_{n'\geq 0}\sum_{m'\geq 0}\Lambda\left(\sum_{j=0}^{P-1} c_{2j+1}\begin{pmatrix} j \\ n' \end{pmatrix}\begin{pmatrix} P-j-1 \\ m' \end{pmatrix} k^{P-n'-m'-1}\right)(\partial^{m'} f)(\partial^{n'} Dg)+$$

$$+\sum_{n'\geq 0}\sum_{m'\geq 0}\Lambda(-1)^{|f|+1}\left(\sum_{j=0}^{P-1} c_{2j+1}\begin{pmatrix} j \\ n' \end{pmatrix}\begin{pmatrix} P-j-1 \\ m' \end{pmatrix} k^{P-n'-m'-1}\right)(\partial^{m'} Df)(\partial^{n'} g)+$$

$$+\sum_{n\geq 0}\sum_{m\geq 0}\left(\sum_{j=0}^{P-1} c_{2j+1}\begin{pmatrix} j \\ n \end{pmatrix}\begin{pmatrix} P-j-1 \\ m \end{pmatrix} k^{P-n-m-1}\right)(\partial^m Df)(\partial^n Dg) =$$

$$= \sum_{i=0}^{P} c_{2i}(\partial^{P-i} f)(\partial^i g) + \sum_{j=0}^{P-1} c_{2j-1}(\partial^{P-j-1} Df)(\partial^j Dg)$$

where $\Lambda = \hat{\zeta} + \theta k$. From this, we must have for every m, n subjected to $0 \leq n \leq i \leq P-m$ and $j \leq P-m'$.

$$\sum_{i=0}^{P} c_{2i}\begin{pmatrix} i \\ n \end{pmatrix}\begin{pmatrix} P-i \\ m \end{pmatrix} k^{P-n-m} = c_{2n}\delta_{P-n-m} \tag{4.2}$$

Also due to the fact that the supergauge invariance has to be obeyed for every f and g we must have $c_{2j+1} = 0$ The discrete equation (4.2) was solved in [17]. Its general solution is given by:

$$\begin{aligned} c_{2i} &= c_0(-1)^i\begin{pmatrix} P \\ i \end{pmatrix} \\ c_{2j+1} &= 0 \end{aligned} \tag{4.3}$$

In the case of $N = 2P+1$ we proceed in a similar manner and we obtain the following system:

$$\sum_{i=0}^{P} c_{2i}\begin{pmatrix} i \\ n \end{pmatrix}\begin{pmatrix} P-i \\ m \end{pmatrix} k^{P-n-m} = c_{2n}\delta_{P-n-m} \tag{4.4}$$

$$\sum_{j=0}^{P} c_{2j+1} \begin{pmatrix} j \\ n \end{pmatrix} \begin{pmatrix} P-j-1 \\ m \end{pmatrix} k^{P-n-m-1} = c_{2n+1}\delta_{P-n-m-1} \tag{4.5}$$

$$(-1)^{|f|} c_{2i} + c_{2i+1} = 0 \tag{4.6}$$

This system has the following solution:

$$c_{2i} = c_0(-1)^i \begin{pmatrix} P \\ i \end{pmatrix}$$

$$c_{2i+1} = c_0(-1)^{i+1+|f|} \begin{pmatrix} P \\ i \end{pmatrix} \tag{4.7}$$

The relations (4.3), (4.7) can be written in a compact form as

$$c_i = c_0(-1)^{i|f|+\frac{i(i+1)}{2}} \begin{bmatrix} N \\ i \end{bmatrix}.$$

and the theorem is proved. We mention that the super-bilinear operator proposed by McArthur and Yung [15] is a particular case of the above super-Hirota operator.

We shall note the bilinear operator as

$$S_N(f,g) := \mathbf{S}_x^N f \bullet g$$

In the Appendix 1 we list several simple properties of this super-Hirota operator.

V. BILINEAR SUSY KDV-TYPE EQUATIONS

SUSY KdV of Manin-Radul-Mathieu

In order to use the super-bilinear operators defined above we shall consider the following nonlinear substitution for the superfield:

$$\Phi(t,x,\theta) = 2D^3 \log \tau(t,x,\theta) \tag{5.1}$$

where τ is an even superfield. Introducing in SUSY KdV (3.2) and integrating with respect to x we obtain the following

$$2D\partial_t \log\tau + 3\{(2D^3\log\tau)(2\partial_x^2\log\tau)\} + 2D^7\log\tau = 0$$

Using the properties (6.4), (6.5), (6.6) we find

$$\frac{\mathbf{S}_x\mathbf{D}_t\tau\bullet\tau}{\tau^2} + 3\left(\frac{\mathbf{S}_x^3\tau\bullet\tau}{\tau^2}\frac{\mathbf{D}_x^2\tau\bullet\tau}{\tau^2}\right) + 2\partial_x^3(\frac{D\tau}{\tau}) = 0 \tag{5.2}$$

Now we are using the following property of the *classical* Hirota operator [19]

$$\partial_x^3(\frac{a}{b}) = \frac{\mathbf{D}_x^3 a\bullet b}{b^2} - 3\frac{\mathbf{D}_x a\bullet b}{b^2}\frac{\mathbf{D}_x^2 b\bullet b}{b^2}$$

In our case $a = D\tau$ and $b = \tau$. Accordingly

$$2\partial_x^3\left(\frac{D\tau}{\tau}\right) = \frac{\mathbf{S}_x^7\tau\bullet\tau}{\tau^2} - 3\left(\frac{\mathbf{S}_x^3\tau\bullet\tau}{\tau^2}\frac{\mathbf{D}_x^2\tau\bullet\tau}{\tau^2}\right)$$

Plugging into (5.2) we find the following super-bilinear form

$$(\mathbf{S}_x\mathbf{D}_t + \mathbf{S}_x^7)\tau\bullet\tau = 0, \tag{5.3}$$

which is equivalent with the form found by McArthur and Yung [15]

$$\mathbf{S}_x(\mathbf{D}_t + \mathbf{D}_x^3)\tau\bullet\tau = 0. \tag{5.4}$$

In order to find the super-soliton solutions we are going to use the classical perturbative method namely the series

$$\tau = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \epsilon^3 f^{(3)} + ... \tag{5.5}$$

where $f^{(i)}$ are even functions. Equating the power of ϵ we find:

- for ϵ,

$$D(\partial_t + \partial_x^3)f^{(1)} = 0 \tag{5.6}$$

- for ϵ^2,

$$2D(\partial_t + \partial_x^3)f^{(2)} = -\mathbf{S}_x(\mathbf{D}_t + \mathbf{D}_x^3)f^{(1)} \bullet f^{(1)}. \tag{5.7}$$

- for ϵ^3,

$$D(\partial_t + \partial_x^3)f^{(3)} = -\mathbf{S}_x(\mathbf{D}_t + \mathbf{D}_x^3)f^{(1)} \bullet f^{(2)}. \tag{5.8}$$

- for ϵ^4,

$$2D(\partial_t + \partial_x^3)f^{(4)} = -2\mathbf{S}_x(\mathbf{D}_t + \mathbf{D}_x^3)f^{(1)} \bullet f^{(3)} - \mathbf{S}_x(\mathbf{D}_t + \mathbf{D}_x^3)f^{(2)} \bullet f^{(2)} \tag{5.9}$$

  and so on.

Now if we take $f^{(1)} = e^{kx-k^3t+\theta\hat{\zeta}+\eta^{(0)}}$ the equation (5.6) is satisfied, $f^{(2)} = 0$, $f^{(3)} = 0$.... and the series (5.5) truncates. So, the 1 supersoliton solution is given by

$$\tau^{(1)} = 1 + e^{kx-k^3t+\theta\hat{\zeta}+\eta^{(0)}} \tag{5.10}$$

for every k and $\hat{\zeta}$.

Introducing in the super-bilinear equation the 1 soliton solution $F = 1 + \exp(kx + \omega t + \hat{\zeta}\theta)$ one finds the *dispersion supermanifold* equation:

$$P(k, \omega, \hat{\zeta}) \equiv (\hat{\zeta} + \theta k)(\omega + k^3) = 0$$

which imposes $\omega = -k^3$ for every $\hat{\zeta}$.

In order to find 2 super-soliton solution we take $f^{(1)} = e^{\eta_1} + e^{\eta_2}$ where $\eta_i = k_i x - k_i^3 t + \theta\hat{\zeta}_i$ The equation (5.7) becomes

$$2D(\partial_t + \partial_x^3)f^{(2)} = 6k_1k_2(k_1 - k_2)[(\hat{\zeta}_1 - \hat{\zeta}_2) + \theta(k_1 - k_2)]e^{\eta_1+\eta_2} \tag{5.11}$$

Taking into account that τ is a Grassmann even function, $f^{(2)}$ must be also even. Accordingly, the general solution of (5.11) has the form

$$f^{(2)} = \left[m_{12}(k_1, k_2, \hat{\zeta}_1, \hat{\zeta}_2) + \theta\hat{n}_{12}(k_1, k_2, \hat{\zeta}_1, \hat{\zeta}_2)\right] e^{\eta_1+\eta_2},$$

m_{12} and $\hat{n}_{12}$ being Grassmann valued even and odd functions. Due to the fact that we have two Grassmann parameters ($\hat{\zeta}_1$ and $\hat{\zeta}_2$) the *most general* expressions for m_{12} and $\hat{n}_{12}$ are given by,

$$m_{12} = (k_1 - k_2)^2/(k_1 + k_2)^2 + \gamma(k_1, k_2)\hat{\zeta}_1\hat{\zeta}_2$$

$$\hat{n}_{12} = a(k_1, k_2)\hat{\zeta}_1 + b(k_1, k_2)\hat{\zeta}_2$$

The above forms for m_{12} and $\hat{n}_{12}$ could also be obtained by expanding in power series of $\hat{\zeta}_1$ and $\hat{\zeta}_2$ and taking into account that in the bosonic limit ($\hat{\zeta}_i \to 0$), $m_{12} \to (k_1-k_2)^2/(k_1+k_2)^2$ (ordinary KdV interaction term) and $\hat{n}_{12} \to 0$

Introducing in the equation (5.11) we find

$$f^{(2)} = \left[\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + 2\frac{k_1 - k_2}{(k_1 + k_2)^2}\hat{\zeta}_1\hat{\zeta}_2 + 2\theta\frac{(k_1 - k_2)(k_1\hat{\zeta}_2 - k_2\hat{\zeta}_1)}{(k_1 + k_2)^2}\right] e^{\eta_1+\eta_2}$$

Introducing the above forms for $f^{(1)}$ and $f^{(2)}$ in (5.8) one obtains $f^{(3)} = 0$ and then $f^{(4)} = 0$, $f^{(5)} = 0$... i.e. the series truncates. So the 2 super-soliton solution is given by;

$$\tau^{(2)} = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \tag{5.12}$$

where

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + 2\frac{k_1 - k_2}{(k_1 + k_2)^2}\hat{\zeta}_1\hat{\zeta}_2 + 2\theta\frac{(k_1 - k_2)(k_1\hat{\zeta}_2 - k_2\hat{\zeta}_1)}{(k_1 + k_2)^2} \tag{5.13}$$

One can easily see that the classical general procedure for finding the interaction term given by the equation (2.7)

$$A_{12}P(k_1 + k_2) + P(k_1 - k_2) = 0$$

does not work in the SUSY case.

In order to find 3 supersoliton solution we consider

$$f^{(1)} = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}$$

and the equation for $f^{(3)}$ (5.8)becomes:

$$D(\partial_t + \partial_x^3)f^{(3)} = -(\mathbf{S}_x\mathbf{D}_t + \mathbf{S}_x^7)(e^{\eta_1} \bullet A_{23}e^{\eta_2+\eta_3} + e^{\eta_2} \bullet A_{13}e^{\eta_1+\eta_3} + e^{\eta_3} \bullet A_{12}e^{\eta_1+\eta_2}) \tag{5.14}$$

The solution $f^{(3)}$ of this equation (which is very complicated) *does not* cancel the right hand side of the equation (5.9). So, $f^{(4)}$ is not zero and the series (5.5) cannot be truncated. Accordingly the SUSY KdV equation does not have 3 super-soliton solution in the standard form for arbitrary k_i's and $\hat{\zeta}_i$'s. This seems a quite strange paradox, because SUSY KdV is integrable in the sense of Lax.

Anyway, imposing the constraint $k_i\hat{\zeta}_j = k_j\hat{\zeta}_i$ for every i and j it is easy to prove (see Appendix 2) that SUSY KdV possesess the following N-soliton solution

$$\tau^{(N)} = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i\eta_i + \sum_{i<j} A_{ij}\mu_i\mu_j\right), \tag{5.15}$$

where

$$\eta_i = k_i x - k_i^3 t + \theta\hat{\zeta}_i + \eta_i^{(0)}$$
$$\exp A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$
$$k_i\hat{\zeta}_j = k_j\hat{\zeta}_i$$

Solutions with constraints on parameters have been found also by Liu and Manas [10], using SUSY Darboux transformation.

SUSY KdV-B

This situation is completely different in the SUSY KdV-B case

$$\Phi_t + D^6\Phi + 6D^2\Phi D\Phi = 0. \tag{5.16}$$

With the same nonlinear substitution

$$\Phi = 2D^3 \log F$$

we obtain the ordinary form

$$(\mathbf{D}_t\mathbf{D}_x + \mathbf{D}_x^4)F \bullet F = 0$$

which *has* N-super-soliton solution (2.5) the fermionic contribution being only an additive phase i.e. $\eta_i = k_i x - k_i^3 t + \hat{\zeta}_i\theta$.

SUSY Sawada-Kotera-Ramani

For $\mathcal{N}=1$ SUSY Sawada-Kotera-Ramani (3.5) using the nonlinear substitution,

$$\Phi = 2D^3 \log \tau(t,x,\theta)$$

we shall find the following super-bilinear form:

$$(\mathbf{S}_x \mathbf{D}_t + \mathbf{S}_x^{11})\tau \bullet \tau = 0 \tag{5.17}$$

In a similar way we find the 2 super-soliton solution

$$\tau^{(2)} = 1 + e^{\eta_1} + e^{\eta_2} + \left[\left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \frac{k_1^2 - k_1 k_2 + k_2^2}{k_1^2 + k_1 k_2 + k_2^2} + 2 \frac{k_1 - k_2}{(k_1 + k_2)^2} \frac{k_1^2 - k_1 k_2 + k_2^2}{k_1^2 + k_1 k_2 + k_2^2} \hat{\zeta}_1 \hat{\zeta}_2 \right] \times$$

$$\times (1 + 2\theta \frac{k_2 \hat{\zeta}_1 - k_1 \hat{\zeta}_2}{k_1 - k_2}) e^{\eta_1 + \eta_2}$$

Also with the same constraint $k_i \hat{\zeta}_j = k_j \hat{\zeta}_i$ we find

$$\tau^{(N)} = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} A_{ij} \mu_i \mu_j \right), \tag{5.18}$$

where

$$\eta_i = k_i x - k_i^5 t + \theta \hat{\zeta}_i + \eta_i^{(0)}$$

$$\exp A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2 \frac{k_i^2 - k_i k_j + k_j^2}{k_i^2 + k_i k_j + k_j^2}$$

SUSY Hirota-Satsuma

For $\mathcal{N}=1$ SUSY Hirota-Satsuma equation using the nonlinear substitution:

$$\Phi = 2D \log \tau(t,x,\theta)$$

one obtains the super-bilinear form:

$$(\mathbf{S}_x^5 \mathbf{D}_t - \mathbf{S}_x^3 - \mathbf{S}_x \mathbf{D}_t)\tau \bullet \tau = 0 \tag{5.19}$$

The 2 supersoliton solution is given by:

$$\tau^{(2)} = 1 + e^{\eta_1} + e^{\eta_2} + \left[\left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 M_{12} + 2 \frac{k_1 - k_2}{(k_1 + k_2)^2} M_{12} \hat{\zeta}_1 \hat{\zeta}_2 \right] (1 + 2\theta \frac{k_2 \hat{\zeta}_1 - k_1 \hat{\zeta}_2}{k_1 - k_2}) e^{\eta_1 + \eta_2}$$

where

$$M_{12} = \frac{(k_1 - k_2)^2 + k_1 k_2 [(k_1 - k_2)^2 - (k_1^2 - 1)(k_2^2 - 1)]}{(k_1 - k_2)^2 - k_1 k_2 [(k_1 - k_2)^2 - (k_1^2 - 1)(k_2^2 - 1)]}$$

and $\eta_i = k_i x - k_i t/(k_i^2 - 1) + \hat{\zeta}_i \theta$. With the constraints $k_i \hat{\zeta}_j = k_j \hat{\zeta}_i$ the equation admits also N soliton solution with

$$A_{ij} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 M_{12}.$$

SUSY Burgers

An interesting case is the SUSY extension of the Burgers equation, namely

$$\Phi_t + \Phi D \Phi_x + \Phi_{xx} = 0$$

It is welknown that the classical Burgers equation can be linearized via Cole-Hopf transform. If we try the nonlinear substitution (which is the natural supersymmetrization of the Cole-Hopf transform):

$$\Phi = 2D \log \tau(t, x, \theta)$$

we find:

$$\mathbf{S}_x \mathbf{D}_t \tau \bullet \tau + 2\mathbf{D}_x^2 D \tau \bullet \tau = 0.$$

This form is not super-gauge invariant. Accordingly we are forced to use two functions for substitution, namely

$$\Phi = \frac{\hat{G}(t, x, \theta)}{F(t, x, \theta)}$$

where $\hat{G}$ is an odd Grassmann function and F is an even one. Using the relation (6.7) from Appendix 1 we obtain the following gauge-invariant super-bilinear form

$$(\mathbf{D}_t + \mathbf{D}_x^2) \hat{G} \bullet F = 0$$

$$\mathbf{D}_x^2 F \bullet F = \mathbf{S}_x^3 \hat{G} \bullet F$$

This system admits the following 1 super-shock solution:

$$\hat{G} = 2(\hat{\zeta} + \theta k) e^{(kx - k^2 t + \hat{\zeta}\theta)}, \quad F = 1 + e^{(kx - k^2 t + \hat{\zeta}\theta)}$$

We can ask ourselves if it is possible to obtain super-bilinear forms for SUSY equations of the nonlinear Klein-Gordon type. In fact the SUSY sine-Gordon equation(3.9) can be written in the following form:

$$[D_T, D_X]\Phi(T, X, \theta, \theta_t) = 2i \sin \Phi(T, X, \theta, \theta_t) \tag{5.20}$$

where we have introduced the light-cone variables $X := i(t-x)/2$, $T := i(t+x)/2$, and $\theta := \theta_1$, $\theta_2 := -\theta_t$. Covariant derivatives are $D_X := \partial_\theta + \theta\partial_X$, $D_T := \partial_{\theta_t} + \theta_t\partial_T$ and the square braket means the commutator. Using the nonlinear substitution (G and F are even functions)

$$\Phi = 2i \log\left(\frac{G}{F}\right),$$

we find the following quadrilinear expression

$$2i\{F^2(G[D_T, D_X]G - [D_T G, D_X G]) - G^2(F[D_T, D_X]F - [D_T F, D_X F])\} = F^4 - G^4$$

It is easy to see that the bilinear operator

$$\mathbf{S}_{XT}\tau \bullet \tau := \tau[D_T, D_X]\tau - [D_T\tau, D_X\tau]$$

is super-gauge invariant with respect to the super-gauge

$$e^\Theta := e^{(kx+\omega t+\theta\hat{\zeta}+\theta_t\hat{\Omega}+liniars)}.$$

Accordingly we can choose the following super-bilinear form, formally the same with standard sine-Gordon equation,

$$\begin{aligned} \mathbf{S}_{XT}G \bullet G &= \frac{1}{2i}(F^2 - G^2) \\ \mathbf{S}_{XT}F \bullet F &= \frac{1}{2i}(G^2 - F^2) \end{aligned} \tag{5.21}$$

but, it is not clear how to compute the super-kink solutions.

¿From these examples it seems that gauge-invariance is a useful concept for bilinear formalism in the supersymmetric case, though there is no deep reason for that. As a consequence we was able to bilinearize two supersymmetric equations of KdV type and the SUSY

sine-Gordon. The case of SUSY versions for mKdV, NLS, KP etc. requires further investigation because it seems that *only certain supersymmetric extensions are super-bilinearizable.* Although we do not know if the SUSY extension of Sawada-Kotera proposed above are integrable in the sense of Lax, it admits super-bilinear form and only 2 super-soliton solution for arbitrary choice of solitary waves. The strange fact is that SUSY KDV equation of Mathieu which is known to be Lax integrable does not admit $Ngeq3$ supersoliton solution in the canonical form. Probably a singularity analysis implemented on the super-bilinear form will reveal the connection between Hirota-integrability and Lax-integrability.

VI. APPENDIX 1

In this section we are going to list several properties of the super-Hirota bilinear operator which are useful in deriving bilinear forms.

$$\mathbf{S}_x^{2N} f \bullet g = \mathbf{D}_x^N f \bullet g \tag{6.1}$$

$$\mathbf{S}_x^{2N+1} e^{\eta_1} \bullet e^{\eta_2} = [\hat{\zeta}_1 - \hat{\zeta}_2 + \theta(k_1 - k_2)](k_1 - k_2)^N e^{\eta_1+\eta_2} \tag{6.2}$$

$$\mathbf{S}_x^{2N+1} 1 \bullet e^{\eta_1} = (-1)^{N+1}(\hat{\zeta} + \theta k)k^N e^{\eta} = (-1)^{N+1}\mathbf{S}_x^{2N+1} e^{\eta} \bullet 1 \tag{6.3}$$

where $\eta_i = k_i x + \theta\hat{\zeta}_i$ and $\hat{\zeta}_i$ are odd Grassmann numbers.

$$2D \log \tau = \frac{D\tau}{\tau} \tag{6.4}$$

$$2D^3 \log \tau = \frac{\mathbf{S}_x^3 \tau \bullet \tau}{\tau^2} \tag{6.5}$$

$$2D\partial_t \log \tau = \frac{\mathbf{S}_x \mathbf{D}_t \tau \bullet \tau}{\tau^2} \tag{6.6}$$

where τ is an even Grassmann function Moreover if G and F are Grassmann functions (with F even) then

$$D^3(\frac{G}{F}) = \frac{\mathbf{S}_x^3 G \bullet F}{F^2} - (-1)^{|G|}\frac{G}{F}\frac{\mathbf{S}_x^3 F \bullet F}{F^2} \tag{6.7}$$

VII. APPENDIX 2

In this section we are going to sketch the proof of the formula for N supersoliton solution for bilinear SUSY KdV equation. We are rely on the proof of Hirota for ordinary N-soliton solution in the case of KdV [21], [23]. Thus, introducing the expression of the N supersoliton solution

$$\tau^{(N)} = \sum_{\mu=0,1} \exp(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} A_{ij}\mu_i\mu_j),$$
$$\eta_i = k_i x - k_i^3 t + \theta\hat{\zeta}_i + \eta_i^{(0)}$$
$$\exp A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$
$$k_i\hat{\zeta}_j = k_j\hat{\zeta}_i$$

into the super bilinear form

$$(\mathbf{S}_x\mathbf{D}_t + \mathbf{S}_x^7)\tau \bullet \tau = 0,$$

and taking into account the properties (6.1) and (6.2) we find

$$\sum_{\mu=0,1}\sum_{\nu=0,1}\{(\sum_{i=1}^{N}(\mu_i-\nu_i)\Lambda_i)((-)\sum_{i=1}^{N}(\mu_i-\nu_i)k_i^3) + (\sum_{i=1}^{N}(\mu_i-\nu_i)\Lambda_i)(\sum_{i=1}^{N}(\mu_i-\nu_i)k_i)^3\}\times$$
$$\times \exp(\sum_{i=1}^{N}(\mu_i+\nu_i)\eta_i + \sum_{i<j}(\mu_i\mu_j+\nu_i\nu_j)A_{ij}) = 0$$

where $\Lambda = \hat{\zeta}_i + \theta k_i$. Since $\mu_i, \nu_i = 0, 1$ it is clear that we only have exponential terms of the form

$$\exp(\sum_{i=1}^{n}\eta_i + \sum_{i=n+1}^{m} 2\eta_i), 0 \le n \le N, n \le m \le N$$

Next we show that the coefficient of this general exponential term is zero, the coefficient being given by, [21] [23]:

$$\Delta = \sum_{\sigma=0,1}\{(-\sum_{i=1}^{N}\sigma_i\Lambda_i)(\sum_{i=1}^{N}\sigma_i k_i^3) + (\sum_{i=1}^{N}\sigma_i\Lambda_i)(\sum_{i=1}^{N}\sigma_i k_i)^3\}\prod_{i<j}^{n}(\sigma_i k_i - \sigma_j k_j)^2 \tag{7.1}$$

where $\sigma_i = \mu_i - \nu_i$. We do not go into details because the procedure of deriving the above coefficient goes absolutely in the same way as in the case of ordinary KdV equation. This is due to the fact that the interaction term A_{ij} is the same for KdV and SUSY KdV.

Because $\Lambda = \hat{\zeta}_i + \theta k_i$ the coefficient Δ becomes on the components

$$\Delta \equiv \Delta_0 + \Delta_1 = \sum_{\sigma=0,1} \{(-\sum_{i=1}^{N} \sigma_i \hat{\zeta}_i)(\sum_{i=1}^{N} \sigma_i k_i^3) + (\sum_{i=1}^{N} \sigma_i \hat{\zeta}_i)(\sum_{i=1}^{N} \sigma_i k_i)^3\} \prod_{i<j}^{n} (\sigma_i k_i - \sigma_j k_j)^2 + \tag{7.2}$$

$$+\theta \sum_{\sigma=0,1} \{(-\sum_{i=1}^{N} \sigma_i k_i)(\sum_{i=1}^{N} \sigma_i k_i^3) + (\sum_{i=1}^{N} \sigma_i k_i)(\sum_{i=1}^{N} \sigma_i k_i)^3\} \prod_{i<j}^{n} (\sigma_i k_i - \sigma_j k_j)^2$$

Hirota proved that the second term is zero [21], [23]. The first term is also zero because, using the property $\hat{\zeta}_i k_j = \hat{\zeta}_j k_i$, it can be written as

$$\Delta_0 = \frac{\hat{\zeta}_m}{k_m} \Delta_1 = 0$$

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